Imperial College London

Lecture 3

Time-domain analysis: Zero-input Response

(Lathi 2.1-2.2)

Peter Cheung
Department of Electrical & Electronic Engineering
Imperial College London

URL: www.ee.imperial.ac.uk/pcheung/teaching/ee2_signals
E-mail: p.cheung@imperial.ac.uk

Zero-input response basics

- Remember that for a Linear System

Total response = zero-input response + zero-state response

- In this lecture, we will focus on a linear system's **zero-input response**, $y_0(t)$, which is the solution of the system equation when input $x(t)$ = 0.

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N)y(t) = (b_{N-M}D^M + b_{N-M+1}D^{M-1} + \cdots + b_{N-1}D + b_N)x(t)$$

$$\Rightarrow \quad Q(D)y(t) = P(D)x(t)$$

$$\Rightarrow \quad Q(D)y_0(t) = 0$$

$$\Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N)y_0(t) = 0$$

L2.2 p152

General Solution to the zero-input response equation(1)

- From maths course on differential equations, we may solve the equation:

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1}D + a_N)y_0(t) = 0 \quad \ldots\ldots\ldots \ (3.1)$$

by letting $y_0(t) = ce^{\lambda t}$, where **c and λ are constants**

- Then:

$$Dy_0(t) = \frac{dy_0}{dt} = c\lambda e^{\lambda t}$$

$$D^2 y_0(t) = \frac{d^2 y_0}{dt^2} = c\lambda^2 e^{\lambda t}$$

$$\vdots$$

$$D^N y_0(t) = \frac{d^N y_0}{dt^N} = c\lambda^N e^{\lambda t}$$

Substitute into (3.1)

L2.2 p153

General Solution to the zero-input response equation(2)

- We get:

$$c(\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)e^{\lambda t} = 0$$

$$\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N = 0 \quad \ldots\ldots\ldots \ (3.1)$$

- This is identical to the polynomial Q(D) **with λ replacing D**, i.e.

$$Q(\lambda) = 0$$

- We can now express Q(λ) in **factorized form**:

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N) = 0 \quad \ldots\ldots\ldots \ (3.2)$$

- Therefore **λ has N solutions**: λ_1, λ_2, …. , λ_N, assuming that all λ_i are distinct.

L2.2 p152

General Solution to the zero-input response equation(3)

- Therefore, equation (3.1): $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0$

  has **N possible solutions**: $c_1 e^{\lambda_1 t},\ c_2 e^{\lambda_2 t},\ \ldots,\ c_N e^{\lambda_N t}$

  where $c_1, c_2, \ldots, c_N$ are arbitrary constants.

- It can be shown that the **general solution** is the sum of all these terms:

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

- In order to determine the N arbitrary constants, we need to have **N constraints** (i.e. initial or boundary or auxiliary conditions).

L2.2 p154

Characteristic Polynomial of a system

- Q(λ) is called the **characteristic polynomial** of the system
- Q(λ) = 0 is the **characteristic equation** of the system
- The **roots** to the characteristic equation Q(λ) = 0, i.e. $\lambda_1, \lambda_2, \ldots, \lambda_N$, are extremely important.
- They are called by different names:
  - Characteristic values
  - **Eigenvalues**
  - **Natural frequencies**
- The exponentials $e^{\lambda_i t} (i = 1, 2, \ldots, n)$ are the **characteristic modes** (also known as **natural modes**) of the system

Characteristics modes determine the system's behaviour

L2.2 p154

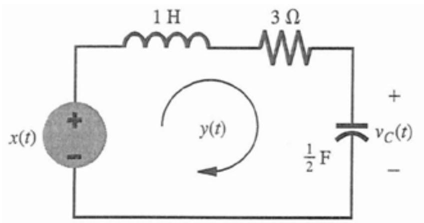
Example 1 (1)

Find $y_0(t)$, the zero-input component of the response, for a LTI system described by the following differential equation:

$$(D^2 + 3D + 2) y(t) = Dx(t)$$

when the initial conditions are

$$y_0(0) = 0, \quad \dot{y}_0(0) = -5.$$



- For zero-input response, we want to find the solution to:

$$(D^2 + 3D + 2) y_0(t) = 0$$

- The **characteristic equation** for this system is therefore:

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

- The **characteristic roots** are therefore $\lambda_1 = -1$ and $\lambda_2 = -2$.
- The **zero-input response** is

$$y_0(t) = c_1 e^{-t} + c_2 e^{-2t}$$

L2.2 p155

Example 1 (2)

- To find the two unknowns c1 and c2, we use the **initial conditions**

$$y_0(0) = 0, \quad \dot{y}_0(0) = -5.$$

- This yields to two simultaneous equations:

$$0 = c_1 + c_2$$
$$-5 = -c_1 - 2c_2$$

- Solving this gives:

$$c_1 = -5 \qquad c_2 = 5$$

- Therefore, the zero-input response of y(t) is given by:

$$y_0(t) = -5e^{-t} + 5e^{-2t}$$

L2.2 p156

Repeated Characteristic Roots

- The discussions so far assume that all characteristic roots are **distinct**. If there are **repeated roots**, the form of the solution is modified.
- The solution of the equation:

$$(D-\lambda)^2 y_0(t) = 0$$

is given by:

$$y_0(t) = (c_1 + c_2 t)e^{\lambda t}$$

- In general, the characteristic modes for the differential equation:

$$(D-\lambda)^r y_0(t) = 0$$

are:

$$e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \ldots, t^{r-1}e^{\lambda t}$$

- The solution for $y_0(t)$ is

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda t}$$

L2.2 p154

Example 2

Find $y_0(t)$, the zero-input component of the response for a LTI system described by the following differential equation: $(D^2 + 6D + 9) = (3D+5)x(t)$

when the initial conditions are $y_0(0) = 3, \ \dot{y}_0(0) = -7$.

- The **characteristic polynomial** for this system is:

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$$

- The **repeated roots** are therefore λ_1 = -3 and λ_2 = -3.
- The **zero-input response** is $y_0(t) = (c_1 + c_2 t)e^{-3t}$
- Now, determine the constants using the initial conditions gives $c_1 = 3$ and $c_2 = 2$.
- Therefore: $y_0(t) = (3 + 2t)e^{-3t} \qquad t \geq 0$

L2.2 p156

Complex Characteristic Roots

- Solutions of the characteristic equation may result in **complex roots**.
- For real (i.e. physically realizable) systems, all complex roots must occur in **conjugate pairs**. In other words, the coefficients of the characteristic polynomial Q(λ) are real.
- In other words, if $\alpha + j\beta$ is a root, then there **must exists** the root $\alpha - j\beta$.
- The zero-input response corresponding to this pair of conjugate roots is:

$$y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}$$

- For a real system, the response $y_0(t)$ must also be real. This is possible only if c_1 and c_2 are conjugates too.
- Let $c_1 = \frac{c}{2}e^{j\theta} \quad \text{and} \quad c_2 = \frac{c}{2}e^{-j\theta}$
- This gives

$$\begin{aligned} y_0(t) &= \frac{c}{2}e^{j\theta}e^{(\alpha+j\beta)t} + \frac{c}{2}e^{-j\theta}e^{(\alpha-j\beta)t} \\ &= \frac{c}{2}e^{\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right] \\ &= ce^{\alpha t}\cos(\beta t + \theta) \end{aligned}$$

L2.2 p155

Example 3 (1)

Find $y_0(t)$, the zero-input component of the response for a LTI system described by the following differential equation: $(D^2 + 4D + 40) = (D+2)x(t)$

when the initial conditions are $y_0(0) = 2, \ \dot{y}_0(0) = 16.78$.

- The characteristic polynomial for this system is:

$$\begin{aligned} \lambda^2 + 4\lambda + 40 &= (\lambda^2 + 4\lambda + 4) + 36 = (\lambda + 2)^2 + (6)^2 \\ &= (\lambda + 2 - j6)(\lambda + 2 + j6) \end{aligned}$$

- The complex roots are therefore $\lambda_1 = -2 + j6$ and $\lambda_2 = -2 - j6$
- The zero-input response in real form is (α = -2, β = 6)

$$y_0(t) = ce^{-2t}\cos(6t + \theta) \qquad \text{…. (12.1)}$$

L2.2 p157

Example 3 (1)

- To find the constants c and θ, we use the initial conditions $y_0(0) = 2, \quad \dot{y}_0(0) = 16.78$.
- Differentiating equation (12.1) $y_0(t) = ce^{-2t}\cos(6t+\theta)$ gives:

$$\dot{y}_0(t) = -2ce^{-2t}\cos(6t+\theta) - 6ce^{-2t}\sin(6t+\theta)$$

- Using the initial conditions, we obtain:

$$2 = c\cos\theta$$
$$16.78 = -2c\cos\theta - 6c\sin\theta$$

- This reduces to:

$$c\cos\theta = 2$$
$$c\sin\theta = -3.463$$

- Hence

$$c^2 = (2)^2 + (-3.464)^2 = 16 \Longrightarrow c = 4 \qquad \theta = \tan^{-1}\left(\frac{-3.463}{2}\right) = -\frac{\pi}{3}$$

- Finally, the solution is

$$y_0(t) = 4e^{-2t}\cos\left(6t - \frac{\pi}{3}\right)$$

L2.2 p157

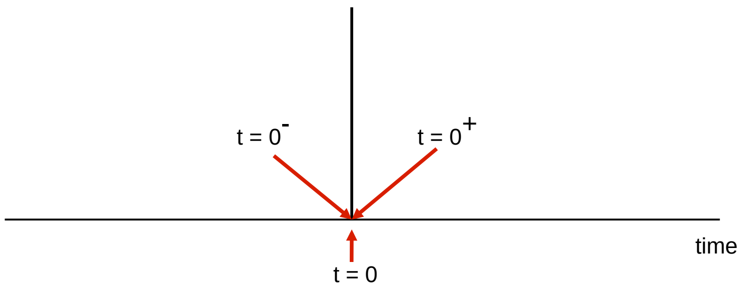
Comments on Auxiliary conditions

- Why do we need **auxiliary** (or **boundary**) **conditions** in order to solve for the zero-input response?
- **Differential** operation is **not invertible** because information is lost.
- To get $y(t)$ from dy/dt, one **extra piece of information** such as $y(0)$ is needed.
- Similarly, if we need to determine $y(t)$ from d^2y/dt^2, we need 2 pieces of information.
- In general, to determine $y(t)$ uniquely from its Nth derivative, we need N additional constraints.
- These constraints are called **auxiliary conditions**.
- When these conditions are given at $t = 0$, they are **initial conditions**.

L2.2 p162

The meaning of 0^- and 0^+

- There are subtle differences between time t = 0 exactly, time just before t=0, i.e. $t = 0^-$ and time just AFTER t=0, i.e. $t = 0^+$.
- At $t = 0^-$ the total response y(t) consists SOLELY of the zero-input component $y_0(t)$.
- However, applying an input x(t) at t=0, while not affecting $y_0(t)$, in general WILL affect y(t) (because input is now no longer zero).



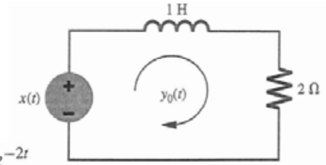
Insights into Zero-input Behaviour

- Assume (a mechanical) system is **initially at rest**.
- Now **disturb it momentarily**, then remove the disturbance (now it is zero-input), the system will not come back to rest instantaneously.
- In generally, it will go back to rest over a period of time, and only through some special type of motion that is **characteristic of the system**.
- Such response must be sustained without any external source (because the disturbance has been removed).
- In fact the system uses a **linear combination** of the characteristic modes to come back to the rest position while satisfying some boundary (or initial) conditions.

L2.2-1 p163

An example

- This example demonstrates that any combination of characteristic modes can be sustained by the system with no external input.
- Consider this RL circuit:



- The loop equation is: $(D+2)y(t) = x(t)$
- It has a single characteristic root λ = -2, and the characteristic mode is e^{-2t}
- Therefore, the loop current equation is $y(t) = ce^{-2t}$
- Now, let us compute the input x(t) required to sustain this loop current:

$$\begin{aligned} x(t) &= L\frac{dy}{dt} + Ry(t) \\ &= \frac{d}{dt}(ce^{-2t}) + 2ce^{-2t} \\ &= -2ce^{-2t} + 2ce^{-2t} \\ &= 0 \end{aligned}$$

The loop current is sustained by the RL circuit on its own without any external input.

The Resonance Behaviour

- Any signal consisting of a system's characteristic mode is **sustained by the system** on its own.
- In other words, the system offers NO obstacle to such signals.
- It is like asking an alcoholic to be a whisky taster.
- Driving a system with an input of the form of the characteristic mode will cause **resonance behaviour**.
- Demonstration:

Relating this lecture to other courses

- Zero-input response is very important to understanding control systems. However, the 2nd year Control course will approach the subject from a different point of view.
- You should also have come across some of these concepts last year in Circuit Analysis course, but not from a "black box" system point of view.
- Ideas in this lecture is essential for deep understanding of the next two lectures on impulse response and on convolution, both you have touched on in your first year in the Communications course.